

# The path-integral analysis of an associative memory model storing an infinite number of finite limit cycles

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## Abstract

An exact solution of the transient dynamics of an associative memory model storing an infinite number of limit cycles with  $l$  finite steps is shown by means of the path-integral analysis. Assuming the Maxwell construction ansatz, we have succeeded in deriving the stationary state equations of the order parameters from the macroscopic recursive equations with respect to the finite-step sequence processing model which has retarded self-interactions. We have also derived the stationary state equations by means of the signal-to-noise analysis (SCSNA). The signal-to-noise analysis must assume that crosstalk noise of an input to spins obeys a Gaussian distribution. On the other hand, the path-integral method does not require such a Gaussian approximation of crosstalk noise. We have found that both the signal-to-noise analysis and the path-integral analysis give completely the same result with respect to the stationary state in the case where the dynamics is deterministic, when we assume the Maxwell construction ansatz. We have shown the dependence of the storage capacity ( $\alpha_c$ ) on the number of patterns per one limit cycle ( $l$ ). At  $l = 1$ , the storage capacity is  $\alpha_c = 0.138$  as in the Hopfield model. The storage capacity monotonically increases with the number of steps, and converges to  $\alpha_c = 0.269$  at  $l \simeq 10$ . The original properties of the finite-step sequence processing model appear as long as the number of steps of the limit cycle has order  $l = O(1)$ .

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## 1. Introduction

In recent years, theories that can analyse the transient dynamics have been discussed for systems with frustrations, especially a correlation-type associative memory [1–11]. Düring *et al* presented a path-integral method for an infinite-step sequence processing model and analysed the properties of the stationary state [12]. By using Düring *et al*'s analysis, Kawamura and Okada succeeded in deriving an exact macroscopic description of the transient dynamics [13]. The transient dynamics can be analysed not only by using the path-integral method [14, 15] but also by using the signal-to-noise analysis, e.g. statistical neurodynamics [16, 17]. The signal-to-noise analysis is an approximation theory in which crosstalk noise obeys a Gaussian distribution. On the other hand, the path-integral method does not require such a Gaussian approximation of crosstalk noise. However surprisingly, the macroscopic equations of the exact solution given by means of the path-integral method are completely equivalent to those of the signal-to-noise analysis with respect to this model.

It has turned out that the infinite-step sequence processing model can be more easily analysed than the Hopfield model even if it is necessary to treat the dynamical process directly. The reason for this is as follows. The retrieval state of the infinite-step sequence processing model has no equilibrium state. Therefore, the correlations of the system are not very complex. Since the Hopfield model takes the same states repeatedly, its statistical properties are more complex than the infinite-step sequence processing model. Gardner *et al* analysed the transient dynamics of the Hopfield model by using the path-integral method in the case where the dynamics is deterministic [18]. They obtained the macroscopic equations of the transient dynamics at time step  $t$  using  $O(t^2)$  macroscopic variables and also obtained the macroscopic equations of the equilibrium state from the transient dynamics. These are equivalent to replica symmetric (RS) solutions given by using the replica method [18]. Recently in the Hopfield model, Bolle *et al* compared the transient dynamics of the path-integral method with those of the signal-to-noise analysis only for a few time steps in the dynamics [19]. They have pointed out that the signal-to-noise analysis is exact up to time step 3 and inexact to step 4 or beyond.

In order to discuss the relation between the path-integral method and the signal-to-noise analysis in more detail, we analyse a finite-step sequence processing model, which includes the Hopfield model and the infinite-step sequence processing model in special cases. In the finite-step sequence processing model, the steady states of the system become limit cycles. Since the finite-step sequence processing model can store limit cycles in the dynamics, the properties of the system are periodic and dynamic essentially like the infinite-step sequence processing model. Moreover, the statistical properties of the finite-step sequence processing model are more complex than the infinite-step one. Since the period of the limit cycle is finite, the network takes the same states repeatedly. Namely, the finite-step sequence processing model has the theoretical difficulties of both the Hopfield model and the infinite-step sequence processing model. From this point of view, it would be very interesting to theoretically discuss the properties of the finite-step sequence processing model.

In this paper, we have exactly derived the transient dynamics of macroscopic recursive equations with respect to the finite-step sequence processing model by means of the path-integral analysis. Until now, only in the infinite-step sequence processing model, which has no self-interactions, Düring *et al* derived the stationary state equations of the order parameters by using the path-integral analysis [12]. The transient dynamics of various disordered systems can also be analysed by using the path-integral method. Therefore, it is important to derive stationary state equations of the order parameters from the macroscopic recursive equations. Assuming the Maxwell construction ansatz, we have succeeded in deriving the stationary

state equations from the macroscopic recursive equations with respect to the model which has self-interactions, i.e. the finite-step sequence processing model.

We also analysed the finite-step sequence processing model by means of the signal-to-noise analysis (SCSNA). The stationary state equations given by the path-integral analysis are equivalent to those of the signal-to-noise analysis. This result corresponds to the fact that the replica method and the signal-to-noise analysis give completely equivalent results in the stationary state analysis of the Hopfield model. Namely, the transient dynamics given by the signal-to-noise analysis gives an exact solution in both the stationary state and the first few time steps in the dynamics.

**2. Definitions**

Let us consider a system storing an infinite number of limit cycles with  $l$  finite steps. The system consists of  $N$  Ising-type spins (or neurons)  $\sigma_i = \pm 1$ . We consider the case where  $N \rightarrow \infty$ . The spins update the state synchronously with the probability

$$\text{Prob}[\sigma_i(t + 1) = -\sigma_i(t)] = \frac{1}{2}[1 - \sigma_i(t) \tanh \beta h_i(t)] \tag{1}$$

$$h_i(t) = \sum_{j=1}^N J_{ij} \sigma_j(t) + \theta_i(t) \tag{2}$$

where  $\beta$  is the inverse temperature,  $\beta = 1/T$ . When the temperature is  $T = 0$ , the updating rule of the state is deterministic. The term  $\theta_i(t)$  is a time-dependent external field which is introduced in order to define a response function. The interaction  $J_{ij}$  stores  $p$  random patterns  $\xi^{v,\mu} = (\xi_1^{v,\mu}, \dots, \xi_N^{v,\mu})^T$  so as to retrieve the patterns as

$$\xi^{v,1} \rightarrow \xi^{v,2} \rightarrow \dots \rightarrow \xi^{v,l} \rightarrow \xi^{v,1} \tag{3}$$

sequentially for any  $\mu$ th limit cycle. For instance, the entries of the interaction matrix  $\mathbf{J} = (J_{ij})$  are given by

$$J_{ij} = \frac{1}{N} \sum_{v=1}^{p/l} \sum_{\mu=1}^l \xi_i^{v,\mu+1} \xi_j^{v,\mu} \tag{4}$$

where the pattern index  $\mu$  are understood to be taken modulo  $l$ . Since the number of limit cycles is  $p/l$ , the total number of stored patterns is  $p$ . The number of stored patterns  $p$  is given by  $p = \alpha N$ , where  $\alpha$  is called the loading rate. In our analysis, the number of steps for each limit cycle  $l$  is kept finite. Each component of the patterns is assumed to be an independent random variable that takes a value of either +1 or -1 according to the probability

$$\text{Prob}[\xi_i^{v,\mu} = \pm 1] = \frac{1}{2}. \tag{5}$$

For the subsequent analysis, the matrix  $\mathbf{J}$  is represented as

$$\mathbf{J} = \frac{1}{N} \xi^T \mathbf{S} \xi \tag{6}$$

where the  $p \times N$  matrix  $\xi$  is defined as

$$\xi = (\xi^{1,1} \dots \xi^{1,l} \xi^{2,1} \dots \xi^{2,l} \dots \xi^{p/l,1} \dots \xi^{p/l,l})^T \tag{7}$$

and the  $p \times p$  matrix  $\mathbf{S}$  is defined as

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}' & \dots & 0 \\ & \ddots & \\ 0 & \dots & \mathbf{S}' \end{pmatrix} \tag{8}$$

and the  $l \times l$  matrix is defined as  $\mathbf{S}' = (S'_{\mu\nu}) = (\delta_{\mu,(v+1) \bmod l})$ . When  $l = 1$ , i.e.  $\mathbf{S} = \mathbf{1}$  ( $\mathbf{1}$  is the unit matrix), the system is equivalent to the Hopfield model.

### 3. Path-integral analysis

Düring *et al* discussed the sequential associative memory model by means of the path-integral analysis [12]. In this section, we introduce macroscopic state equations for the model with a finite temperature  $T \geq 0$ , according to their paper.

In order to analyse the transient dynamics, the generating function  $Z[\psi]$  is defined as

$$Z[\psi] = \sum_{\sigma(0), \dots, \sigma(t)} p[\sigma(0), \dots, \sigma(t)] \exp \left( -i \sum_{s < t} \sigma(s) \cdot \psi(s) \right) \quad (9)$$

where  $\psi = (\psi(0), \dots, \psi(t-1))^T$  and the state  $\sigma(s) = (\sigma_1(s), \dots, \sigma_N(s))^T$  denotes the state of the spins at time  $s$ . The probability  $p[\sigma(0), \dots, \sigma(t)]$  denotes the probability of taking the path from initial state  $\sigma(0)$  to state  $\sigma(t)$  at time  $t$  through  $\sigma(1), \sigma(2), \dots, \sigma(t-1)$ . As (9) shows, the generating functional entails the summation of all  $2^{(t+1)N}$  paths which the system can take from time 0 to  $t$ . One can obtain all the relevant order parameters, i.e. the overlap  $m(s)$ , the correlation function  $C(s, s')$  and the response function  $G(s, s')$ , by calculating the appropriate derivatives of the above functional and letting  $\psi$  tend to  $\mathbf{0}$  afterwards as follows:

$$m(s) = i \lim_{\psi \rightarrow \mathbf{0}} \frac{1}{N} \sum_{i=1}^N \xi_i^{1,s} \frac{\partial Z[\psi]}{\partial \psi_i(s)} \quad (10)$$

$$C(s, s') = - \lim_{\psi \rightarrow \mathbf{0}} \frac{1}{N} \sum_{i=1}^N \frac{\partial^2 Z[\psi]}{\partial \psi_i(s) \partial \psi_i(s')} \quad (11)$$

$$G(s, s') = i \lim_{\psi \rightarrow \mathbf{0}} \frac{1}{N} \sum_{i=1}^N \frac{\partial^2 Z[\psi]}{\partial \psi_i(s) \partial \theta_i(s')} \quad (12)$$

Using the assumption of self-averaging, we replace the generating functional  $Z[\psi]$  with its disorder-averaged generating functional  $\bar{Z}[\psi]$ . Evaluating the averaged generating function  $\bar{Z}[\psi]$  through the saddle-point method, we obtain the following saddle-point equations for the order parameters of (10)–(12) in the thermodynamical limit, i.e.  $N \rightarrow \infty$  (see appendix A).

$$m(s) = \langle\langle \xi \sigma(s) \rangle\rangle \quad (13)$$

$$C(s, s') = \langle\langle \sigma(s) \sigma(s') \rangle\rangle \quad (14)$$

$$G(s, s') = \frac{\partial \langle\langle \sigma(s) \rangle\rangle}{\partial \theta(s')} \quad (15)$$

The average over the effective path measure is given by

$$\langle\langle g(\sigma, v) \rangle\rangle = \left\langle \int \mathcal{D}v \operatorname{Tr}_{\sigma} g(\sigma, v) p[\sigma(0)] \prod_{s=1}^t \frac{1}{2} [1 + \sigma(s) \tanh \beta h(\sigma, v, s-1)] \right\rangle_{\xi} \quad (16)$$

$$\mathcal{D}v \equiv \frac{d\mathbf{v} e^{-\frac{1}{2} \mathbf{v} \cdot \mathbf{R}^{-1} \mathbf{v}}}{\sqrt{(2\pi)^l |\mathbf{R}|}} \quad (17)$$

$$\operatorname{Tr}_{\sigma} \equiv \sum_{\sigma(0), \dots, \sigma(t) \in \{-1, 1\}} \quad (18)$$

$$h(\sigma, v, s) = \xi^{s+1} m(s) + \theta(s) + \sqrt{\alpha} v(s) + (\mathbf{\Gamma} \sigma)(s) \quad (19)$$

$$\mathbf{R} = \sum_{a=0}^{l-1} \sum_{m, n \geq 0} \mathbf{G}^{ml+a} C(\mathbf{G}^{\dagger})^{nl+a} \quad (20)$$

$$\Gamma = \frac{\alpha}{l} \sum_{\mu=0}^{l-1} e^{2\pi i \mu / l} [\mathbf{1} - e^{2\pi i \mu / l} \mathbf{G}]^{-1} \tag{21}$$

with  $\Gamma = \hat{K}^\dagger, \hat{Q} = -\frac{1}{2}\alpha i \hat{R}$  and  $p[\sigma(0)] = \frac{1}{2}[1 + \sigma(0)m(0)]$  which is the initial spin probability. The operator  $\langle \cdot \rangle_\xi$  denotes the average over the condensed patterns. The term  $(\Gamma\sigma)(s)$  denotes the  $s$ th element of the vector  $\Gamma\sigma$ . The vectors  $\sigma$  and  $v$  denote  $\sigma = \{\sigma(0), \dots, \sigma(t)\}$  and  $v = \{v(0), \dots, v(t-1)\}$ , respectively. Equations (13)–(21) entirely describe the dynamics of the system. The term  $\prod_{s=1}^t \frac{1}{2}[1 + \sigma(s) \tanh \beta h(\sigma, v, s-1)]$  in (16) cannot be factorized with respect to spin variables at different times. Calculation of the spin summation of (16) requires an exponential time  $O(e^t)$  at time  $t$ . In the infinite-step sequence processing model, the local field  $h(\sigma, v, s)$  depends on only spin variables at time  $s$  [12]. Therefore the term  $\prod_{s=1}^t \frac{1}{2}[1 + \sigma(s) \tanh \beta h(\sigma, v, s-1)]$  can be factorized, so the spin summations can be taken easily.

#### 4. The stationary state

In this section, we inspect time-translation invariant solutions of our macroscopic equations (13)–(15) for the deterministic dynamics, i.e.  $\beta \rightarrow \infty (T = 0)$ . The time-translation invariant solutions will describe motion on a macroscopic limit cycle:

$$\begin{cases} m(s) = m \\ C(s, s') = C(s - s') \\ G(s, s') = G(s - s') \end{cases} \tag{22}$$

with  $\theta(s) = \theta$ . Now, we disregard the transient states. Note that the condition of (22) includes an unspoken condition that the transient states are disregarded. Therefore, we put that the dynamics is already in the stationary state at time  $s = 0$  under this assumption. In the zero noise limit, i.e.  $T = 0 (\beta \rightarrow \infty)$ , the dynamics becomes deterministic. Therefore, we also assume that the system takes a fixed path as

$$\sigma(s+l) = \sigma(s) \tag{23}$$

for any time  $s \geq 0$ . The path which the system takes after time  $s \geq 0$  can be described as

$$\sigma(s) = \eta_s \tag{24}$$

by only  $l$  constants  $\eta_0, \dots, \eta_{l-1} \in \{-1, 1\}$ . The pattern index  $s$  of the constants  $\eta_s$  is understood to be taken modulo  $l$ . Note that it is not necessary to calculate these constants  $\{\eta_s\}$  explicitly. When the variable transformation

$$\chi(s) = \eta_s \sigma(s) \tag{25}$$

is carried out to spin variables  $\sigma(s)$ , the transformed spin variables  $\chi(s)$  take the same value for any time  $s$ , i.e.  $\chi(s) = \chi(s')$  for any  $s, s'$ . Equation (16) means the expectation of  $g(\sigma, v)$  with respect to the path probability. In the zero noise limit, equation (16) becomes

$$\langle\langle g(\sigma, v) \rangle\rangle = \left\langle \int \mathcal{D}v \operatorname{Tr}_\sigma g(\sigma, v) p[\sigma(0)] \prod_{s=1}^t \delta_{\sigma(s), \operatorname{sgn} h(\sigma, v, s-1)} \right\rangle_\xi \tag{26}$$

When the spin variables have periodicity as  $\sigma(s+l) = \sigma(s)$ , the Gaussian random fields are also deterministic as  $v(s+l) = v(s)$  (see appendix B). For any function  $\phi(\{\sigma(s)\})$  and any constants  $c_0, \dots, c_t \in \{-1, 1\}$ , the following identity holds:

$$\sum_{\sigma(0), \dots, \sigma(t) \in \{-1, 1\}} \phi(\sigma(0), \dots, \sigma(t)) = \sum_{\sigma(0), \dots, \sigma(t) \in \{-1, 1\}} \phi(c_0 \sigma(0), \dots, c_t \sigma(t)). \tag{27}$$

Applying (27) to (26) and substituting (25), we obtain

$$\begin{aligned}
 \langle\langle g(\boldsymbol{\sigma}, \mathbf{v}) \rangle\rangle &= \left\langle \int \mathcal{D}\mathbf{v} \operatorname{Tr}_{\boldsymbol{\sigma}} g(\{c_0\boldsymbol{\sigma}(0), \dots, c_t\boldsymbol{\sigma}(t)\}, \mathbf{v}) p[c_0\boldsymbol{\sigma}(0)] \right. \\
 &\quad \left. \times \prod_{s=1}^t \delta_{c_s\boldsymbol{\sigma}(s), \operatorname{sgn} h(\{c_0\boldsymbol{\sigma}(0), \dots, c_t\boldsymbol{\sigma}(t)\}, \mathbf{v}, s-1)} \right\rangle_{\xi} \\
 &= \left\langle \int \mathcal{D}\mathbf{v} \operatorname{Tr}_{\chi} g(\{c_0\eta_0\chi(0), \dots, c_t\eta_t\chi(t)\}, \mathbf{v}) \right. \\
 &\quad \left. \times p[c_0\eta_0\chi(0)] \prod_{s=1}^t \delta_{c_s\eta_s\chi(s), \operatorname{sgn} h(\{c_0\eta_0\chi(0), \dots, c_t\eta_t\chi(t)\}, \mathbf{v}, s-1)} \right\rangle_{\xi} \\
 &= \left\langle \int \mathcal{D}\mathbf{v} \operatorname{Tr}_{\chi} g(\{\chi(0), \dots, \chi(t)\}, \mathbf{v}) p[\chi(0)] \prod_{s=1}^t \delta_{\chi(s), \operatorname{sgn} h(\{\chi(0), \dots, \chi(t)\}, \mathbf{v}, s-1)} \right\rangle_{\xi}
 \end{aligned} \tag{28}$$

with

$$\operatorname{Tr}_{\chi} \equiv \sum_{\eta_0\chi(0), \dots, \eta_t\chi(t) \in \{-1, 1\}} = \sum_{\chi(0), \dots, \chi(t) \in \{-1, 1\}} . \tag{29}$$

In the derivation of (28), we put the constants  $\{c_s\}$  as  $c_s = \eta_s$  for all  $s$ . Generality is kept even if  $c_s = \eta_s$  for all  $s$ . Namely, with respect to the transformed spin variable  $\chi(s)$ , the effective single spin described by (28) is

$$\chi(s) = \operatorname{sgn} h(\{\chi(0), \dots, \chi(t)\}, \mathbf{v}, s-1). \tag{30}$$

The transformed spin variables  $\chi(s)$  are deterministic even if (30) includes the Gaussian random fields  $\mathbf{v}$ , since the Gaussian random fields are deterministic. In order to get rid of the self-interaction, we assume the Maxwell construction ansatz. Using the identity  $\chi(s) = \chi(s')$  for any  $s, s'$  to (30) and applying the Maxwell construction, we get

$$\begin{aligned}
 \chi(s) &= \operatorname{sgn} \left[ \xi^s m(s-1) + \theta(s-1) + \sqrt{\alpha} v(s-1) + \sum_{s' < s} \Gamma(s, s') \chi(s') \right] \\
 &= \operatorname{sgn} \left[ \xi^s m(s-1) + \theta(s-1) + \sqrt{\alpha} v(s-1) + \chi(s) \sum_{s' < s} \Gamma(s, s') \right] \\
 &= \operatorname{sgn} h(\mathbf{v}, s-1)
 \end{aligned} \tag{31}$$

with  $h(\mathbf{v}, s) \equiv \xi^{s+1} m(s) + \theta(s) + \sqrt{\alpha} v(s)$ . Substituting (31) into (28), we obtain

$$\langle\langle g(\boldsymbol{\sigma}, \mathbf{v}) \rangle\rangle = \left\langle \int \mathcal{D}\mathbf{v} \operatorname{Tr}_{\chi} g(\chi, \mathbf{v}) p[\chi(0)] \prod_{s=1}^t \delta_{\chi, \operatorname{sgn} h(\mathbf{v}, s-1)} \right\rangle_{\xi}. \tag{32}$$

Thus, we can get rid of the self-interaction in the single spin problem by using the Maxwell construction in the zero noise limit, i.e.  $T = 0$ . Since (32) can be factorized with respect to the transformed spin variables  $\chi(s)$  at different times, we can easily perform the spin summations. After simple rescalings we arrive at

$$m(s) = \left\langle \xi^s \int \mathcal{D}\mathbf{v} \operatorname{sgn} h(\mathbf{v}, s-1) \right\rangle_{\xi} \tag{33}$$

$$C(s, s') = \delta_{s, s'} + [1 - \delta_{s, s'}] \left\langle \int \mathcal{D}\mathbf{v} [\operatorname{sgn} h(\mathbf{v}, s-1)] [\operatorname{sgn} h(\mathbf{v}, s'-1)] \right\rangle_{\xi} \tag{34}$$

$$G(s, s') = \delta_{s, s'-1} \lim_{\beta \rightarrow \infty} \beta \left\{ 1 - \left\langle \int \mathcal{D}v \tanh^2 \beta h(v, s - 1) \right\rangle_{\xi} \right\}. \tag{35}$$

We now calculate the matrix  $R$  under the condition of (22). Since the matrices  $G$  and  $C$  become Toeplitz matrices (especially if  $C$  is symmetric) under these conditions,  $C$  and  $G$  can be approximately regarded as commuting matrices, i.e.  $CG = GC$ . Therefore, the matrix  $R$  simplifies to

$$\begin{aligned} R &= \sum_{a=0}^{l-1} G^a \left( \sum_{m, n \geq 0} G^{ml} (G^\dagger)^{nl} C \right) (G^\dagger)^a \\ &= \sum_{a=0}^{l-1} G^a ([\mathbf{1} - G^l]^{-1} [\mathbf{1} - (G^\dagger)^l]^{-1} C) (G^\dagger)^a \\ &= [\mathbf{1} - GG^\dagger]^{-1} [\mathbf{1} - (GG^\dagger)^l]^{-1} [\mathbf{1} - G^l]^{-1} [\mathbf{1} - (G^\dagger)^l]^{-1} C \end{aligned} \tag{36}$$

We consider the persistent parts of  $C(\tau)$  and  $R(\tau)$  as  $C(\tau) \rightarrow q$  and  $R(\tau) \rightarrow r$  for  $\tau \rightarrow \infty$  and also consider the non-persistent parts  $\tilde{C}(\tau) \rightarrow 0$  and  $\tilde{R}(\tau) \rightarrow 0$ , i.e.  $C(\tau) = q + \tilde{C}(\tau)$  and  $R(\tau) = r + \tilde{R}(\tau)$ . Upon rewriting  $G(\tau) = \beta \delta_{\tau, 1} [\mathbf{1} - \tilde{q}]$  and  $r = q\rho$  given by (36), we obtained

$$m = \left\langle \xi \int \mathcal{D}z \int \tilde{\mathcal{D}}v \operatorname{sgn} h(v, 0|z) \right\rangle_{\xi} \tag{37}$$

$$q = \left\langle \int \mathcal{D}z \int \tilde{\mathcal{D}}v [\operatorname{sgn} h(v, \tau|z)] [\operatorname{sgn} h(v, 0|z)] \right\rangle_{\xi} \tag{38}$$

$$\tilde{q} = \lim_{\beta \rightarrow \infty} \left\langle \int \mathcal{D}z \int \tilde{\mathcal{D}}v \tanh^2 \beta h(v, 0|z) \right\rangle_{\xi} \tag{39}$$

$$r = q\rho \tag{40}$$

from (33)–(35) with  $\tilde{\mathcal{D}}v \equiv dv e^{-\frac{1}{2}v \cdot \tilde{R}^{-1}v} [(2\pi)^t |\tilde{R}|]^{-1/2}$ ,  $\mathcal{D}z = \frac{dz}{\sqrt{2\pi}} e^{-z^2/2}$  and  $h(v, \tau|z) \equiv \xi^\tau m + \theta + z \sqrt{\alpha q \rho} + \sqrt{\alpha} v(\tau)$ . The matrix  $G$  is given by

$$G = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ G(1) & 0 & 0 & \cdots & 0 \\ 0 & G(1) & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & G(1) & 0 \end{pmatrix}. \tag{41}$$

from (35). Equation (36) can be changed to

$$[\mathbf{1} - (G^\dagger)^l - G^l - (G^\dagger G)^l] [\mathbf{1} - (GG^\dagger)^l]^{-1} [\mathbf{1} - GG^\dagger] R = C. \tag{42}$$

The identity  $GG^\dagger \simeq G(1)^2 \mathbf{1}$  holds in the case where time  $s$  is sufficiently large, so the left-hand side of (42) becomes

$$\begin{aligned} &[\mathbf{1} - (G^\dagger)^l - G^l - (G^\dagger G)^l] [\mathbf{1} - (GG^\dagger)^l]^{-1} [\mathbf{1} - GG^\dagger] R \\ &= \begin{pmatrix} D(0) & D(1) & \cdots & D(t-1) \\ D(1) & D(0) & \cdots & D(t-2) \\ \vdots & \vdots & \ddots & \vdots \\ D(t-1) & D(t-2) & \cdots & D(0) \end{pmatrix} \end{aligned} \tag{43}$$

where

$$D(0) = (1 + g_1^2)g_2R(0) - 2g_1g_2R(1) \quad (44)$$

$$D(s) = (1 + g_1^2)g_2R(s) - g_1g_2[R(s-1) + R(s+1)] \quad (0 < s < t-1) \quad (45)$$

$$D(t-1) = (1 + g_1^2)g_2R(t-1) - 2g_1g_2R(t-2) \quad (46)$$

with  $g_1 \equiv G(1)^l$  and  $g_2 \equiv [1 - G(1)^2]/[1 - G(1)^{2l}]$ . Since  $D$  and  $C$  are symmetric Toeplitz matrices, they can be diagonalized by using the discrete Fourier transformation (see appendix C). The Fourier transformations (or the lattice Green functions)  $\hat{D}_k, \hat{C}_k$  of the matrices  $D, C$  are given by

$$\hat{D}_k \simeq \sum_{\tau=0}^{t-1} \{(1 + g_1^2)g_2R(\tau) - g_1g_2[R(\tau-1) + R(\tau+1)]\} e^{ik\tau} \quad (47)$$

$$\hat{C}_k = \sum_{\tau=0}^{t-1} C(\tau) e^{ik\tau}. \quad (48)$$

For any wave number  $k$ ,  $\hat{D}_k = \hat{C}_k$  holds when  $D = C$ . Taking the limit  $s \rightarrow \infty$  about  $\hat{D}_0 = \hat{C}_0$ , the following relationship is obtained:

$$r = \frac{1 - G(1)^{2l}}{(1 - G(1)^2)(1 - G(1)^l)^2} q. \quad (49)$$

By working out the remaining integrals over  $v$  and setting  $\theta = 0$ , we finally obtain the stationary state equations of the order parameters as follows:

$$m = \operatorname{erf}\left(\frac{m}{\sqrt{2\alpha\rho}}\right) \quad (50)$$

$$U = \sqrt{\frac{2}{\pi\alpha\rho}} e^{-\frac{m^2}{2\alpha\rho}} \quad (51)$$

$$\rho = \frac{1 - U^{2l}}{(1 - U^2)(1 - U^l)^2} \quad (52)$$

with  $q = 1, \tilde{q} = 1$  and  $U \equiv G(1)$  where  $\operatorname{erf}(\cdot)$  denotes the error function defined by  $\operatorname{erf}(x) \equiv \frac{2}{\pi} \int_0^x e^{-u^2} du$ . It turns out that these stationary state equations (50)–(52) given by this exact solution are equivalent to those of the signal-to-noise analysis (see appendix D) [5]. Figure 1 shows the storage capacity  $\alpha_c$  and the number of patterns per one limit cycle  $s$ . Figures 2 and 3 compare the theoretical results and computer simulations for  $l = 3, 7$  (the number of spins is  $N = 3000$ , and the number of iterations is 11). The data points and error bars show the results of the computer simulation. With respect to the computer simulation in figures 2 and 3, the stationary overlaps are defined as  $m(100l)$  and  $m(50l)$ , respectively. It is confirmed that the theoretical results are in good agreement with the computer simulations. Storage capacity monotonically increases from  $\alpha_c = 0.138$  ( $l = 1$ ) with the number of steps  $l$ . In the large  $l$  limit, the storage capacity finally converges to  $\alpha_c = 0.269$ , which coincides with the theoretical result for the infinite-step sequence processing model given by Düring *et al* [12]. The original properties of the finite-step sequence processing model appear as long as the number of steps of a limit cycle  $l$  has order  $l = O(1)$ . In the case where  $l$  has order more than  $O(1)$ , the properties are the same as the properties of the infinite-step sequence processing model.

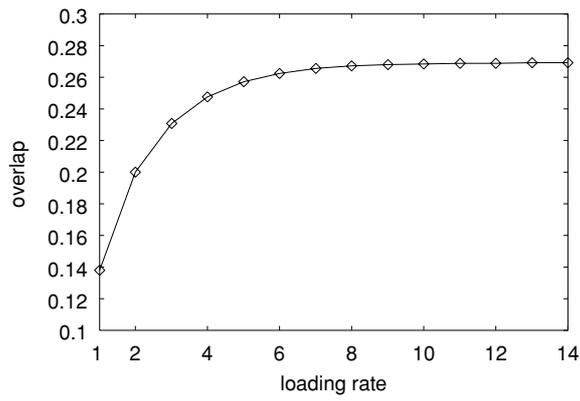


Figure 1. The storage capacity  $\alpha_c$  and the number of patterns per one limit cycle  $s$ .

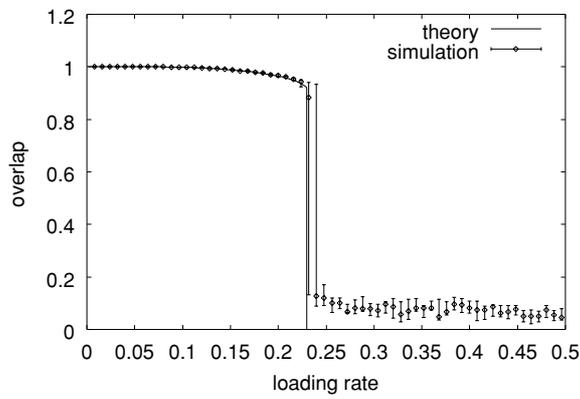


Figure 2. Computer simulations ( $l = 3, N = 3000, 11$  times): the overlap  $m$  and loading rate  $\alpha$ .

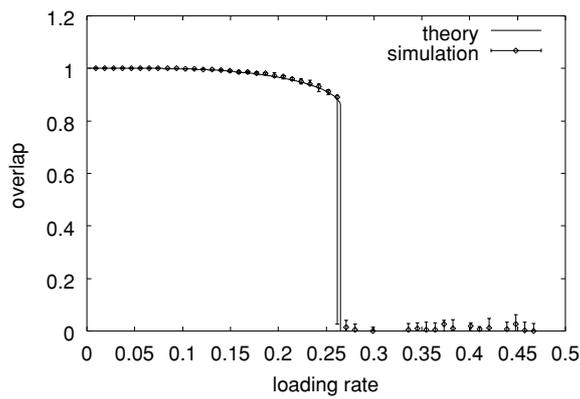


Figure 3. Computer simulations ( $l = 7, N = 3000, 11$  times): the overlap  $m$  and loading rate  $\alpha$ .

## 5. Conclusions

We exactly analysed an associative memory model storing an infinite number of limit cycles with finite steps by means of the path-integral method. In the case where the dynamics is deterministic, the statistical properties are simplified like those of the infinite-step sequence processing model by using the Maxwell construction. We also derived the macroscopic equations for stationary states at  $T = 0$ .

We obtained the dependence of the storage capacity ( $\alpha_c$ ) on the number of patterns per one limit cycle ( $l$ ). At  $l = 1$ , the storage capacity is  $\alpha_c = 0.138$ , as in the Hopfield model. The storage capacity monotonically increases with the number of limit cycles, and converges to  $\alpha_c = 0.269$  at  $l \simeq 10$ . The original properties of the finite-step sequence processing model appear as long as the number of steps of the limit cycle has order  $l = O(1)$ .

In the case where the dynamics is deterministic, we also derived the stationary state equations by using the Maxwell construction. The stationary state equations are equivalent to those of the signal-to-noise analysis. This means that the signal-to-noise analysis applied to the stationary state is exact in spite of including errors in the middle of the transient dynamics in the zero noise limit.

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## Appendix A. Derivation of order parameter equations

Düring *et al* discussed the sequential associative memory model by using the path-integral method [12]. Here we discuss for the model with finite temperature  $T$ , according to their paper. Most of the technical details to derive order parameter equations are almost identical to the paper of Düring *et al* [12].

The generating functional  $\bar{Z}[\psi]$  contains both condensed and non-condensed patterns. We isolate the non-condensed ones by introducing the local field  $\mathbf{h}$  and the variables  $\mathbf{x}, \mathbf{y}$ :

$$1 = \int \frac{d\mathbf{h} d\hat{\mathbf{h}}}{(2\pi)^{Nt}} \prod_i \exp\left(i\hat{h}_i(s) \left[ h_i(s) - \sum_j J_{ij} \sigma_j(s) - \theta_i(s) \right]\right) \quad (\text{A1})$$

$$1 = \int \frac{d\mathbf{x} d\hat{\mathbf{x}}}{(2\pi)^{(p-1)t}} \exp\left(i \sum_{s<t} \sum_{v=1}^{p/l} \sum_{\mu=1(\mu \neq s \text{ at } v=1)}^l \hat{x}_{v\mu}(s) \left[ x_{v\mu}(s) - \frac{1}{\sqrt{N}} \sum_i \xi_i^{v,\mu+1} \hat{h}_i(s) \right]\right) \quad (\text{A2})$$

$$1 = \int \frac{d\mathbf{y} d\hat{\mathbf{y}}}{(2\pi)^{(p-1)t}} \exp\left(i \sum_{s<t} \sum_{v=1}^{p/l} \sum_{\mu=1(\mu \neq s \text{ at } v=1)}^l \hat{y}_{v\mu}(s) \left[ y_{v\mu}(s) - \frac{1}{\sqrt{N}} \sum_i \xi_i^{v,\mu} \sigma_i(s) \right]\right). \quad (\text{A3})$$

We isolate the various relevant macroscopic observables by inserting integrals over appropriate  $\delta$ -functions:

$$1 = \int \frac{d\mathbf{m} d\hat{\mathbf{m}}}{(2\pi/N)^t} \exp\left(i \sum_{s<t} \hat{m}(s) \left[ m(s) - \frac{1}{N} \sum_i \xi_i^{1,s} \sigma_i(s) \right]\right) \quad (\text{A4})$$

$$1 = \int \frac{d\mathbf{k} d\hat{\mathbf{k}}}{(2\pi/N)^t} \exp \left( i \sum_{s < t} \hat{k}(s) \left[ k(s) - \frac{1}{N} \sum_i \xi_i^{1,s+1} \hat{h}_i(s) \right] \right) \quad (\text{A5})$$

$$1 = \int \frac{d\mathbf{q} d\hat{\mathbf{q}}}{(2\pi/N)^{t^2}} \exp \left( i \sum_{s,s' < t} \hat{q}(s,s') \left[ q(s,s') - \frac{1}{N} \sum_i \sigma_i(s) \sigma_i(s') \right] \right) \quad (\text{A6})$$

$$1 = \int \frac{d\mathbf{Q} d\hat{\mathbf{Q}}}{(2\pi/N)^{t^2}} \exp \left( i \sum_{s,s' < t} \hat{Q}(s,s') \left[ Q(s,s') - \frac{1}{N} \sum_i \hat{h}_i(s) \hat{h}_i(s') \right] \right) \quad (\text{A7})$$

$$1 = \int \frac{d\mathbf{K} d\hat{\mathbf{K}}}{(2\pi/N)^{t^2}} \exp \left( i \sum_{s,s' < t} \hat{K}(s,s') \left[ K(s,s') - \frac{1}{N} \sum_i \sigma_i(s) \hat{h}_i(s') \right] \right). \quad (\text{A8})$$

The generating functional which for  $N \rightarrow \infty$  will be dominated by saddle points. We obtain

$$\bar{Z}[\psi] = \int d\mathbf{m} d\hat{\mathbf{m}} d\mathbf{k} d\hat{\mathbf{k}} d\mathbf{q} d\hat{\mathbf{q}} d\mathbf{Q} d\hat{\mathbf{Q}} d\mathbf{K} d\hat{\mathbf{K}} e^{N(\Psi+\Phi+\Omega)+\mathcal{O}(N^{1/2})} \quad (\text{A9})$$

by substituting (A1)–(A8) into (9), where

$$\begin{aligned} \Psi = i \sum_{s < t} [\hat{m}(s)m(s) + \hat{k}(s)k(s) - m(s)k(s)] \\ + i \sum_{s,s' < t} [\hat{q}(s,s')q(s,s') + \hat{Q}(s,s')Q(s,s') + \hat{K}(s,s')K(s,s')] \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} \Phi = \frac{1}{N} \sum_i \ln \text{Tr}_\sigma p_i(\sigma(0)) \int \{d\mathbf{h} d\hat{\mathbf{h}}\} \exp \left( \sum_{s < t} [\beta\sigma(s+1) - \ln 2 \cosh \beta h(s)] \right) \\ \times \exp \left( -i \sum_{s,s' < t} [\hat{q}(s,s')\sigma(s)\sigma(s') + \hat{Q}(s,s')\hat{h}(s)\hat{h}(s') + \hat{K}(s,s')\sigma(s)\hat{h}(s')] \right) \\ \times \exp \left( i \sum_{s < t} \hat{h}(s)[h(s) - \theta_i(s) - \hat{k}(s)\xi_i^{1,s+1}] - i \sum_{s < t} \sigma(s)[\hat{m}(s)\xi_i^{1,s} + \psi_i(s)] \right) \end{aligned} \quad (\text{A11})$$

$$\begin{aligned} \Omega = \frac{1}{N} \ln \int \frac{d\mathbf{u} d\mathbf{v}}{(2\pi)^{(p-t)t}} \exp \left( i \sum_{\mu > t} \sum_{s < t} u_{v,\mu+1}(s)v_{v,\mu}(s) \right) \\ \times \exp \left( -\frac{1}{2} \sum_{v > 1, \mu \geq 1} \sum_{s,s' < t} [u_{v,\mu}(s)Q(s,s')v_{v,\mu}(s') + u_{v,\mu}(s)K(s,s')v_{v,\mu}(s')] \right) \\ \times \exp \left( -\frac{1}{2} \sum_{v > 1, \mu \geq 1} \sum_{s,s' < t} [v_{v,\mu}(s)K(s,s')u_{v,\mu}(s') + v_{v,\mu}(s)q(s,s')v_{v,\mu}(s')] \right) \end{aligned} \quad (\text{A12})$$

with the shorthand notation  $\{d\mathbf{h} d\hat{\mathbf{h}}\} = \prod_i \frac{dh_i(s)d\hat{h}_i(s)}{2\pi}$ .

In the limit  $N \rightarrow \infty$ , the integral (A9) will be dominated by the saddle point of the extensive exponent  $\Psi + \Phi + \Omega$ . The saddle-point equations which are derived by differentiation with respect to integration variables  $\{m, \hat{m}, k, \hat{k}, q, \hat{q}, Q, \hat{Q}, K, \hat{K}\}$  are as follows:

$$\hat{m}(s) = k(s) = 0 \quad (\text{A13})$$

$$Q(s,s') = \hat{q}(s,s') = 0 \quad (\text{A14})$$

$$m(s) = \hat{k}(s) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \langle \sigma(s) \xi_i^{1,s} \rangle_i \quad (\text{A15})$$

$$q(s, s') = C(s, s') = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \langle \sigma(s) \sigma(s') \rangle_i \quad (\text{A16})$$

$$K(s, s') = iG(s, s') = i \lim_{N \rightarrow \infty} \frac{1}{N} \sum_i \frac{\partial \langle \sigma(s) \rangle_i}{\partial \theta_i(s')} \quad (\text{A17})$$

$$\hat{Q}(s, s') = i \lim_{Q \rightarrow 0} \frac{\partial \Omega}{\partial K(s, s')} \Big|_{\text{saddle}} \quad (\text{A18})$$

$$\hat{K}(s, s') = \frac{\partial \Omega}{\partial K(s, s')} \Big|_{\text{saddle}} \quad (\text{A19})$$

where  $f|_{\text{saddle}}$  denotes an evaluation of a function  $f$  at the dominating saddle point,  $\langle \cdot \rangle_i$ , denotes

$$\langle f(\sigma, \mathbf{h}, \hat{\mathbf{h}}) \rangle_i = \left\langle \frac{\text{Tr}_\sigma \int \{d\mathbf{h} d\hat{\mathbf{h}}\} W_i(\sigma, \mathbf{h}, \hat{\mathbf{h}}) f(\sigma, \mathbf{h}, \hat{\mathbf{h}})}{\text{Tr}_\sigma \int \{d\mathbf{h} d\hat{\mathbf{h}}\} W_i(\sigma, \mathbf{h}, \hat{\mathbf{h}})} \right\rangle_\xi \quad (\text{A20})$$

$$\begin{aligned} W_i(\sigma, \mathbf{h}, \hat{\mathbf{h}}) = p_i(\sigma(0)) & \left[ \exp \left( \sum_{s < t} (\beta \sigma(s+1)h(s) - \ln 2 \cosh \beta h(s)) \right) \right. \\ & \times \exp \left( i \sum_{s < t} (\hat{h}(s) \{h(s) - \theta_i(s) - \hat{k}(s) \xi_i^{1,s+1}\} - \theta(s) \hat{m}(s) \xi_i^{1,s}) \right) \\ & \left. \times \exp \left( -i \sum_{s, s' < t} (\hat{q}(s, s') \sigma(s) \sigma(s') + \hat{Q}(s, s') \hat{h}(s) \hat{h}(s') + \hat{K}(s, s') \sigma(s) \hat{h}(s')) \right) \right] \end{aligned} \quad (\text{A21})$$

and  $\langle \cdot \rangle_\xi$  denotes the average over the condensed patterns. We now calculate the right-hand sides of (A18) and (A19). The eigenvalues  $s_\mu$  of a matrix  $\mathbf{S}$  are given by  $s_\mu = e^{2\pi i \mu / l}$  (the multiplicity is  $p/l$ ) from  $|\lambda \mathbf{1} - \mathbf{S}| = |\lambda \mathbf{1} - \mathbf{S}'|^{p/l}$ . Since  $\mathbf{S}'$  is a unitary matrix, i.e.  $\mathbf{S}'^\dagger \mathbf{S}' = \mathbf{1}$ , the matrix  $\mathbf{S}$  is also unitary,  $\mathbf{S}^\dagger \mathbf{S} = \text{diag}(\mathbf{S}'^\dagger, \dots, \mathbf{S}'^\dagger) \text{diag}(\mathbf{S}', \dots, \mathbf{S}') = \text{diag}(\mathbf{S}'^\dagger \mathbf{S}', \dots, \mathbf{S}'^\dagger \mathbf{S}') = \mathbf{1}$ . A  $(\mu, \mu)$ -element of  $(\mathbf{S}^\dagger)^m \mathbf{S}^n$  becomes  $[(\mathbf{S}^\dagger)^m \mathbf{S}^n]_{\mu\mu} = \delta_{mn}$  by using the unitarity of  $\mathbf{S}$ , where  $\delta_{mn}$  denotes Kronecker's delta function. The identity  $(\mathbf{S})^l = (\mathbf{S}^\dagger)^l = \mathbf{S}$  is established because  $(\mathbf{S}')^l = (\mathbf{S}'^\dagger)^l = \mathbf{S}'$ . Therefore, the identity  $[(\mathbf{S}^\dagger)^m \mathbf{S}^n]_{\mu\mu} = \delta_{mn}$  holds for the following value  $(m, n)$ :

$$\begin{cases} m = m'l + a \\ n = n'l + a \end{cases} \quad (\text{A22})$$

with  $m', n' \in \{0, 1, \dots\}$  and  $a \in \{0, \dots, l-1\}$ . The multiplicity of the eigenvalues of the matrix  $\mathbf{S}'$  is 1, so  $\text{rank}(\mathbf{S}' - s_\mu \mathbf{1}) = l - 1$ . Hence,  $\text{rank}(\mathbf{S} - s_\mu \mathbf{1}) = (p/l) \text{rank}(\mathbf{S}' - s_\mu \mathbf{1}) = p - p/l$ . This means that the matrix  $\mathbf{S}$  can be diagonalized by an appropriate non-singular matrix as

$$\text{diag}(\overbrace{s_0, \dots, s_0}^{p/l}, \dots, \overbrace{s_{l-1}, \dots, s_{l-1}}^{p/l}).$$

By working out the saddle-point equation (A18),  $\hat{Q}$  becomes as follows:

$$\hat{Q}(s, s') = -\frac{1}{2} \alpha i \sum_{m, n \geq 0} \lim_{p \rightarrow \infty} \sum_{\mu \leq p} \{(\mathbf{S} \otimes \mathbf{G})^m [\mathbf{1} \otimes \mathbf{C}] (\mathbf{S}^\dagger \otimes \mathbf{G}^\dagger)^n\}_{\mu\mu}(s', s). \quad (\text{A23})$$

Hence  $\hat{Q}$  is given by

$$\hat{Q} = -\frac{1}{2}\alpha i \sum_{a=0}^{l-1} \sum_{m', n' \geq 0} G^{m'l+a} C(G^\dagger)^{n'l+a}. \tag{A24}$$

We define a matrix  $\Gamma = \mathbf{S} \otimes \mathbf{R}$  as having matrix elements  $\Gamma_{\mu\mu'}(s, s') = S_{\mu\mu'} R(s, s')$  for  $\mu, \mu' \in \{1, \dots, p\}$  and  $s, s' \in \{0, \dots, t-1\}$  where  $\mathbf{y} = \Gamma \mathbf{x}$  will operate as  $y_\mu(s) = \sum_{\mu' > t} \sum_{s' < t} S_{\mu\mu'} R(s, s') x_{\mu'}(s')$  for each  $(\mu, s)$ . Equation (A19) reduces to

$$\begin{aligned} \hat{K}(s, s') &= -\frac{1}{2}\alpha \frac{\partial}{\partial G(s, s')} \lim_{p \rightarrow \infty} \frac{1}{p} \{ \ln \det[\mathbf{1} \otimes \mathbf{1} - \mathbf{S}^\dagger \otimes \mathbf{G}^\dagger] + \ln \det[\mathbf{1} \otimes \mathbf{1} - \mathbf{S} \otimes \mathbf{G}] \} \\ &= -\alpha \frac{\partial}{\partial G(s, s')} \lim_{p \rightarrow \infty} \frac{1}{p} \ln \prod_{\mu=0}^{l-1} (\det[\mathbf{1} - e^{2\pi i \mu/l} \mathbf{G}^\dagger])^{p/l} \end{aligned} \tag{A25}$$

$$\hat{K} = \frac{\alpha}{l} \sum_{\mu=0}^{l-1} e^{2\pi i \mu/l} [\mathbf{1} - e^{2\pi i \mu/l} \mathbf{G}^\dagger]^{-1}. \tag{A26}$$

Replacing  $\xi_i^{1,s} \rightarrow \xi^s$ , we obtain the order parameters of (13)–(15).

### Appendix B. The periodicity of the Gaussian random field

Using the time-translation invariant ansatz, i.e.  $C(s, s') = C(s-s')$ ,  $C(s)$  also have periodicity as

$$C(s+l) = C(s) \tag{B1}$$

for any  $s$ , when spin variables  $\sigma(s)$  have periodicity, i.e.  $\sigma(s+l) = \sigma(s)$ . It is confirmed that the vector  $v$  obeys the Gaussian distribution with mean  $\mathbf{0}$  and covariance matrix  $\mathbf{R}$  from equation (16). The covariance matrix  $\mathbf{R}$  of the Gaussian random fields  $v$  is given by (36). Since the matrix  $\mathbf{R}$  is also a symmetric Toeplitz matrix, we can put  $R(s, s') = R(s-s')$  where  $R(s, s')$  are the elements of the matrix  $\mathbf{R}$ . When  $C(s+l) = C(s)$ , we approximately get  $R(s+l) = R(s)$ . The correlation coefficient between crosstalk noise  $v(s)$  and  $v(s+l)$  becomes

$$\text{Corr}(v(s+l), v(s)) = \frac{\text{Cov}(v(s+l), v(s))}{\sqrt{V(v(s+l))V(v(s))}} = \frac{R(s+l, s)}{\sqrt{R(s+l, s+l)R(s, s)}} = 1$$

for any  $s, s'$ . The  $v(s)$  distribution and the  $v(s+l)$  distribution have the same mean and variance, i.e.  $E(v(s)) = E(v(s+l)) = 0$  and  $V(v(s)) = R(s, s) = R(0) = R(s+l, s+l) = V(v(s+l))$ . Therefore, the identity

$$v(s+l) = v(s) \tag{B2}$$

holds for any  $s$ . Hence, the Gaussian random fields  $v(s)$  also have periodicity as  $v(s+l) = v(s)$  when spin variables have periodicity as  $\sigma(s+l) = \sigma(s)$ . Therefore,  $v(s)$  can be considered as quenched noise. After the states  $v(0), \dots, v(l-1)$  occur at random,  $v(s)$  continues taking the same values periodically. Therefore, we can assume that  $v(s)$  is deterministic for a fixed site. The Gaussian random fields  $v(s)$  are only distributed with respect to site index  $i$ .

### Appendix C. The Fourier transformation of the symmetric Toeplitz matrix

Symmetric Toeplitz matrices can be diagonalized by using the discrete Fourier transformation. The Fourier transformation was used to obtain the identity of (49). The Fourier transformation is defined as

$$\hat{\xi}_k = \frac{1}{\sqrt{t}} \sum_j \xi_j e^{-ikj} \quad (\text{C1})$$

and the inverse Fourier transformation is defined as

$$\xi_j = \frac{1}{\sqrt{t}} \sum_k \hat{\xi}_k e^{ikj} \quad (\text{C2})$$

where  $k$  denotes the wave number and its degree of freedom is  $t$ . Each component of  $k$  takes the value  $0, \frac{2}{t}\pi, \frac{4}{t}\pi, \dots, \frac{2(t-1)}{t}\pi$ . The following symmetric Toeplitz matrix can be diagonalized by using the Fourier representation:

$$D = \begin{pmatrix} D_0 & D_1 & \cdots & D_{t-1} \\ D_1 & D_0 & \cdots & D_{t-2} \\ \vdots & \vdots & \ddots & \vdots \\ D_{t-1} & D_{t-2} & \cdots & D_0 \end{pmatrix}. \quad (\text{C3})$$

The Fourier representation of the quadratic form  $\xi^T D \xi$  becomes

$$\begin{aligned} \xi^T D \xi &= \sum_{i=1}^t \sum_{j=1}^t \xi_i D_{|i-j|} \xi_j \\ &= \sum_{\tau=0}^{t-1} \sum_{j=1}^t \xi_j D_\tau \xi_{j-\tau} \\ &= \sum_{\tau} \sum_j \left( \frac{1}{\sqrt{t}} \sum_{k_1} \hat{\xi}_{k_1} e^{ik_1 j} \right) D_\tau \left( \frac{1}{\sqrt{t}} \sum_{k_2} \hat{\xi}_{k_2} e^{ik_2(j-\tau)} \right) \\ &= \sum_k \hat{\xi}_k \left( \sum_{\tau} D_\tau e^{ik\tau} \right) \hat{\xi}_{-k} \end{aligned} \quad (\text{C4})$$

where the index  $j$  of the variables  $\xi_j$  is understood to be taken modulo  $t$ . Therefore, the Fourier transformation of the symmetric Toeplitz matrix  $D = (D_\tau)$  is given by

$$\hat{D}_k = \sum_{\tau} D_\tau e^{ik\tau}. \quad (\text{C5})$$

This transformation is also called the lattice Green function.

### Appendix D. The signal-to-noise analysis of the finite-step sequence processing model

We discuss the stability of limit cycles by means of the signal-to-noise analysis. Let us consider the following deterministic synchronous dynamics:

$$x_i^{t+1} = F \left( \sum_{j=1}^N J_{ij} x_j^t \right) \quad (\text{D1})$$

where the  $x_i^t$  represents the state of the  $i$ th neuron at time  $t$  and  $F(\cdot)$  denotes an output function. The retrieval state converges to some limit cycle with  $l$  steps. We introduced the Poincaré map

to get the states every  $l$  steps in the steady state. The periodic state can be transformed into a stable state by using this map. Hence, we can discuss the properties of a stability of limit cycles.

Let us consider the case of convergence to periodic states of the limit cycle retrieval. We assume  $x_i^t = x_i^{t-l}$  in (D1). The overlap between the  $\mu$ th memory pattern  $\xi^{v\mu}$  of the  $v$ th limit cycle and the network state  $\mathbf{x}$  is defined as

$$m_{v\mu}^t = \frac{1}{N} \sum_{i=1}^N \xi_i^{v\mu} x_i^t. \tag{D2}$$

The dynamics (D1) can be rewritten as

$$x_i^{t+1} = F(h_i^t) \tag{D3}$$

$$h_i^t = \sum_{j=1}^N J_{ij} x_j^t = \sum_{v=1}^{p/l} \sum_{\mu} \xi_i^{v\mu+1} m_{v\mu}^t \tag{D4}$$

where  $h_i^t$  is a local field. Now let us consider the case to retrieve the first limit cycle  $\xi^{1\mu}$ ,  $\mu \in \{1, \dots, l\}$ . We assume the memorized pattern of another limit cycle  $\xi^{v\mu}$ ,  $v \neq 1$ , does not have a finite overlap. We consider retrieval solutions in which  $m_{1\mu}^t \sim O(1)$  and  $m_{v\mu}^t \sim O(1/\sqrt{N})$ ,  $v \geq 2$ . We assume that the components  $x_i$  of the equilibrium state  $\mathbf{x}$  are independent of the unit number  $i$  in the limit  $N \rightarrow \infty$ . It is necessary to assume that the self-averaging property holds too so that the site average can be replaced by an average over the random patterns and random variable  $\mathbf{x}$ . In this situation the overlap  $m_{v\mu}^t$  need not be a random variable.

$$m_{v\mu}^t = \bar{m}_{v\mu}^t + U_t m_{v\mu}^{t-1} \tag{D5}$$

$$\bar{m}_{v\mu}^t = \frac{1}{N} \sum_i \xi_i^{v,\mu} x_i^{t(v\mu)} \tag{D6}$$

$$U_t = \frac{1}{N} \sum_i x_i^{t(v\mu)} \tag{D7}$$

where

$$x_i^{t(v\mu)} = F \left( \sum_{(v'\mu') \neq (v,\mu-1)} \xi^{v'\mu'+1} m_{v'\mu'}^{t-1} \right) \tag{D8}$$

$$x_i^{t(v\mu)} = F' \left( \sum_{(v'\mu') \neq (v,\mu-1)} \xi^{v'\mu'+1} m_{v'\mu'}^{t-1} \right). \tag{D9}$$

By applying (D5) repeatedly, we obtain

$$m_{v\mu}^t = \bar{m}_{v\mu}^t + U_t \bar{m}_{v\mu}^{t-1} + U_t U_{t-1} \bar{m}_{v\mu}^{t-2} + \dots + U_t U_{t-1} \dots U_{t-l+2} \bar{m}_{v\mu}^{t-l+1} + U_t U_{t-1} \dots U_{t-l+1} m_{v\mu}^{t-l}. \tag{D10}$$

Since the retrieval state is assumed to be steady, i.e.  $x_i^{t-l} = x_i^t$ , the identity  $m_{v\mu}^{t-l} = m_{v\mu}^t$  holds. Therefore the overlap becomes

$$m_{v\mu}^t = \left( 1 - \prod_{k=0}^{l-1} U_{t-k} \right)^{-1} \left[ \bar{m}_{v\mu}^t + \sum_{k=1}^{l-1} \prod_{k'=0}^{k-1} U_{t-k'} \bar{m}_{v\mu}^{t-k} \right]. \tag{D11}$$

Substituting (D6) into (D11), we obtain

$$m_{v\mu}^t = \frac{1}{N} \left( 1 - \prod_{k=0}^{l-1} U_{t-k} \right)^{-1} \left[ \sum_{j=1}^N \xi_j^{v\mu} x_j^{t(v\mu)} + \sum_{j=1}^N \xi_j^{v,\mu} x_j^{t(v\mu)} \sum_{k=1}^{l-1} \left( \prod_{k'=0}^{k-1} U_{t-k'} \right) \sum_{j=1}^N \xi_j^{v,\mu-k} x_j^{t-k(v\mu-k)} \right] \quad (\text{D12})$$

Replacing  $\xi_i^{1\mu} \rightarrow \xi_i^\mu$  and  $m_{1\mu}^t \rightarrow m_\mu^t$ , and substituting (D12) into (D4), the local field  $h_i^t$  can be rewritten as

$$h_i^t = \sum_{\mu} \xi_i^{\mu+1} m_\mu^t + \frac{1}{N} \left( 1 - \prod_{k=0}^{l-1} U_{t-k} \right)^{-1} \left[ \sum_{v \geq 2}^{\alpha N/l} \sum_{\mu=1}^l \xi_i^{v\mu+1} \xi_i^{v\mu} x_i^{t(v\mu)} + \sum_{k=0}^{l-1} \left( \prod_{k'=0}^{k-1} U_{t-k'} \right) \sum_{v \geq 2}^{\alpha N/l} \sum_{\mu=1}^l \xi_i^{v\mu+1} \xi_i^{v\mu-k} x_i^{t-k(v\mu-k)} \right] + \frac{1}{N} \left( 1 - \prod_{k=0}^{l-1} U_{t-k} \right)^{-1} \left[ \sum_{v \geq 2}^{\alpha N/l} \sum_{\mu=1}^l \sum_{j \neq i}^N \xi_i^{v\mu+1} \xi_j^{v\mu} x_j^{t(v\mu)} + \sum_{k=0}^{l-1} \left( \prod_{k'=0}^{k-1} U_{t-k'} \right) \sum_{v \geq 2}^{\alpha N/l} \sum_{\mu=1}^l \sum_{j \neq i}^N \xi_i^{v\mu+1} \xi_j^{v\mu-k} x_j^{t-k(v\mu-k)} \right]. \quad (\text{D13})$$

The first term in (D13) is regarded as the signal. The second and the third terms are regarded as the mean and the variance of the crosstalk noise, respectively. In the second term, if the suffix is  $\mu + 1 \equiv \mu - k \pmod{l}$ ,  $\xi_i^{v\mu+1}$  and  $\xi_j^{v\mu-k}$  are represented by the same patterns. Since these terms consist of uncorrelated random variables of the order  $O(1/\sqrt{N})$ , the other terms in the second term can be omitted. The term of  $k = l - 1$  is the only one remaining of the terms with  $k \in \{0, \dots, l - 1\}$ . Hence we can estimate the second term as

$$\frac{1}{N} \left( \prod_{k'=0}^{l-2} U_{t-k'} \right) \left( 1 - \prod_{k'=0}^{l-1} U_{t-k'} \right)^{-1} \sum_{v \geq 2}^{\alpha N/l} \sum_{\mu=1}^l \xi_i^{v\mu+1} \xi_j^{v\mu-l+1} x_j^{t-l+1(v\mu-l+1)} = \alpha \left( \prod_{k'=0}^{l-2} U_{t-k'} \right) \left( 1 - \prod_{k'=0}^{l-1} U_{t-k'} \right)^{-1} x_i^{t+1}. \quad (\text{D14})$$

We assume that the third term is normally distributed with mean 0 and variance  $\sigma_t^2$  because of the independence of  $x_i^{t(v\mu)}$  and  $\xi_i^{v\mu}$ . The variance  $\sigma_t^2$  of the crosstalk noise is estimated as

$$\sigma_t^2 = \alpha \left( 1 - \prod_{k'=0}^{l-1} U_{t-k'} \right)^{-2} \left[ q^t + \sum_{k=1}^{l-1} \left( \prod_{k'=0}^{k-1} U_{t-k'} \right)^2 q^{t-k} \right] \quad (\text{D15})$$

where

$$q^{t-k} = \frac{1}{N} \sum_{j \neq i}^N (x_j^{t-k})^2. \quad (\text{D16})$$

The local field  $h_i$  is obtained by setting  $z_i \sim N(0, 1)$  in (D13),

$$h_i = \sum_{\mu=1}^s \xi_i^{\mu+1} m_\mu^t + \alpha \left( \prod_{k'=0}^{l-2} U_{t-k'} \right) \left( 1 - \prod_{k'=0}^{l-1} U_{t-k'} \right)^{-1} x_i^{t+1} + \sigma_t z_i. \quad (\text{D17})$$

The self-averaging property is assumed. Replacing  $x_i \rightarrow Y$  and  $\xi_i^\mu \rightarrow \xi^\mu$ , we obtain the macroscopic equations as follows:

$$Y^{t+1}(\xi^1, \dots, \xi^l; z) = F \left( \sum_{\mu=1}^l \xi^{\mu+1} m_\mu^t + \Gamma Y^{t+1}(\xi^1, \dots, \xi^l; z) + \sigma_t z \right) \quad (D18)$$

$$m_\mu^{t+1} = \int_{-\infty}^{\infty} Dz \langle \langle \xi^\mu Y^{t+1}(\xi^1, \dots, \xi^l; z) \rangle \rangle \quad (D19)$$

$$q^{t+1} = \int_{-\infty}^{\infty} Dz \langle \langle Y^{t+1}(\xi^1, \dots, \xi^l; z)^2 \rangle \rangle \quad (D20)$$

$$U^{t+1} = \frac{1}{\sigma_t} \int_{-\infty}^{\infty} Dz z \langle \langle Y^{t+1}(\xi^1, \dots, \xi^l; z) \rangle \rangle \quad (D21)$$

$$\sigma_{t+1}^2 = \alpha \left( 1 - \prod_{k'=0}^{l-1} U_{t-k'} \right)^{-2} \left[ q^t + \sum_{k=1}^{l-1} \left( \prod_{k'=0}^{k-1} U_{t-k'} \right) q^{t-k} \right] \quad (D22)$$

$$\Gamma = \alpha \left( \prod_{k'=0}^{l-2} U_{t-k'} \right) \left( 1 - \prod_{k'=0}^{l-1} U_{t-k'} \right)^{-1} \quad (D23)$$

Now let us consider the case that the state of memory retrieval is periodic. We can set

$$m_v^t = m \delta_{vt} \quad q_t = q \quad U_t = U \quad \sigma_t = \sigma \quad (t = 1, \dots, l) \quad (D24)$$

where  $\delta_{vt}$  denotes Kronecker's delta. Finally, we obtain the macroscopic equations as follows:

$$Y(\xi^{t+1}; z) = F(\xi^{t+1} m + \Gamma Y(\xi^{t+1}; z) + \sigma z) \quad (D25)$$

$$m = \int_{-\infty}^{\infty} Dz \langle \langle \xi^{t+1} Y(\xi^{t+1}; z) \rangle \rangle \quad (D26)$$

$$q = \int_{-\infty}^{\infty} Dz \langle \langle Y(\xi^{t+1}; z)^2 \rangle \rangle \quad (D27)$$

$$U = \frac{1}{\sigma} \int_{-\infty}^{\infty} Dz z \langle \langle Y(\xi^{t+1}; z) \rangle \rangle \quad (D28)$$

$$\sigma^2 = \alpha \frac{1 - U^{2l}}{(1 - U^l)^2 (1 - U^2)} q \quad (D29)$$

$$\Gamma = \frac{\alpha U^{l-1}}{1 - U^l}. \quad (D30)$$

Setting  $F(\cdot) = \text{sgn}(\cdot)$  and using the Maxwell rule [2], we obtain (50)–(52) as follows:

$$m = \text{erf} \left( \frac{m}{\sqrt{2\alpha\rho}} \right) \quad (D31)$$

$$U = \sqrt{\frac{2}{\pi\alpha\rho}} e^{-\frac{m^2}{2\alpha\rho}} \quad (D32)$$

$$\rho = \frac{1 - U^{2l}}{(1 - U^2)(1 - U^l)^2} \quad (D33)$$

where  $\sigma^2 = \alpha\rho q$ ,  $q = 1$  and  $\text{sgn}(\cdot)$  denotes the sign function ( $\text{sgn}(x) = 1$  for  $x \geq 0$ ,  $-1$  for  $x < 0$ ). Thus, we find that these stationary state equations of the order parameters given by the signal-to-noise analysis (SCSNA) are equivalent to those of the path-integral analysis (50)–(52).

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